Decomposition of Nonlinear System Dynamics into Multiple Time Scales

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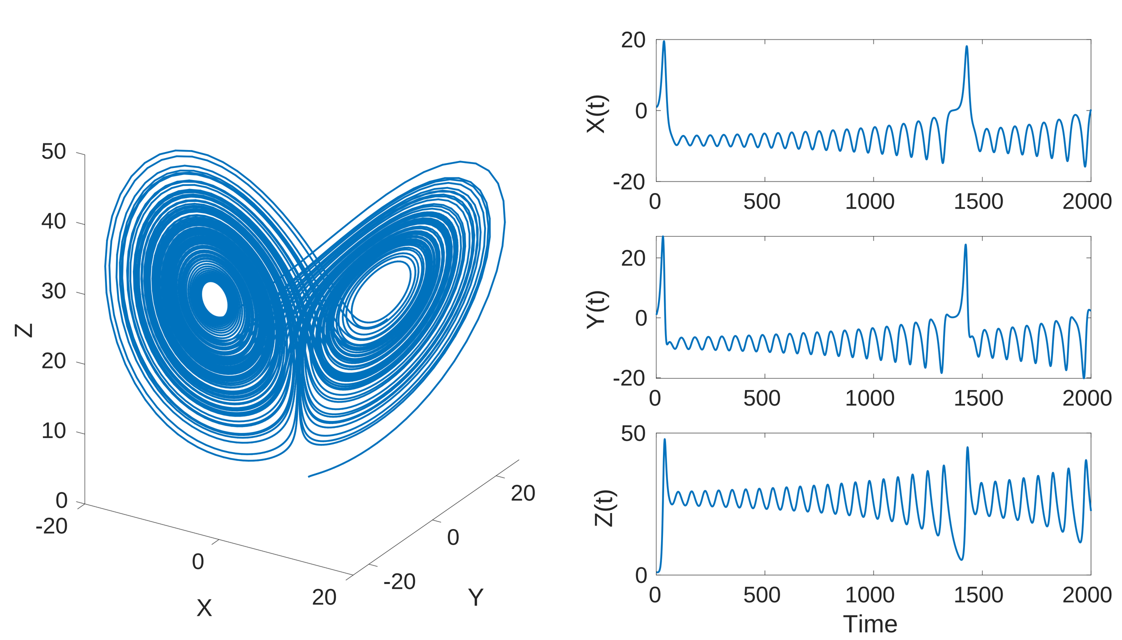
Abstract

We present a novel algorithm which takes two causally-related signals and separates them from their interference. This process is an extension of the convergent cross mapping (CCM) technique developed by Sugihara et al. in 2012. We extend CCM to reconstruct signals while adding implementations of ways to deterministically select optimal tuning parameters. This algorithm is then applied to analyze experimental Hall thruster data, from which we are able to recreate two distinct constituent signals.

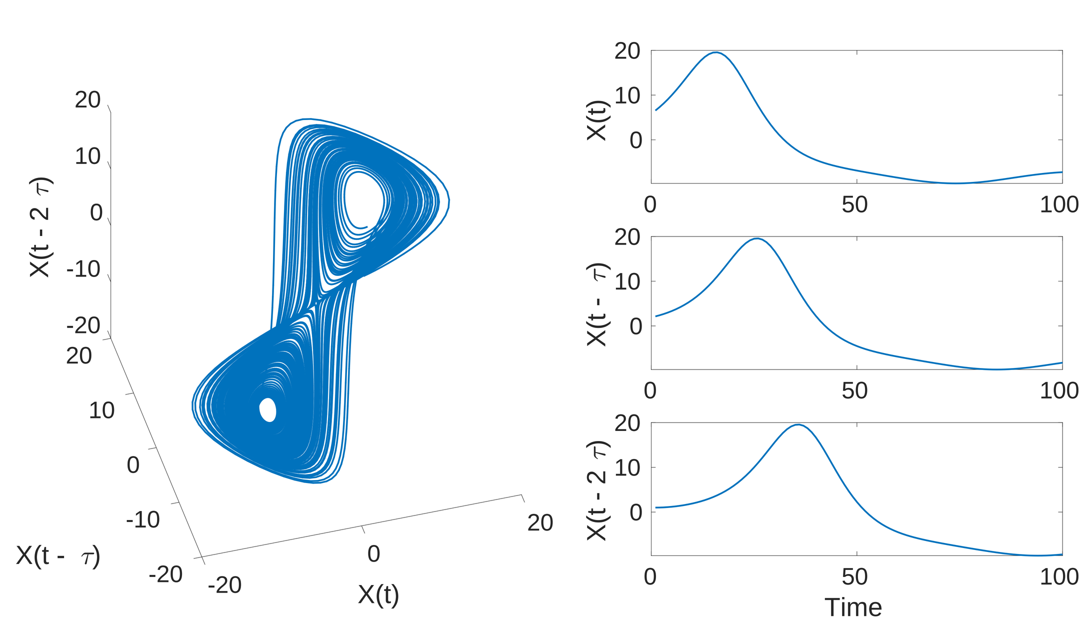
Mathematical Background

Time series representation of the Lorenz system

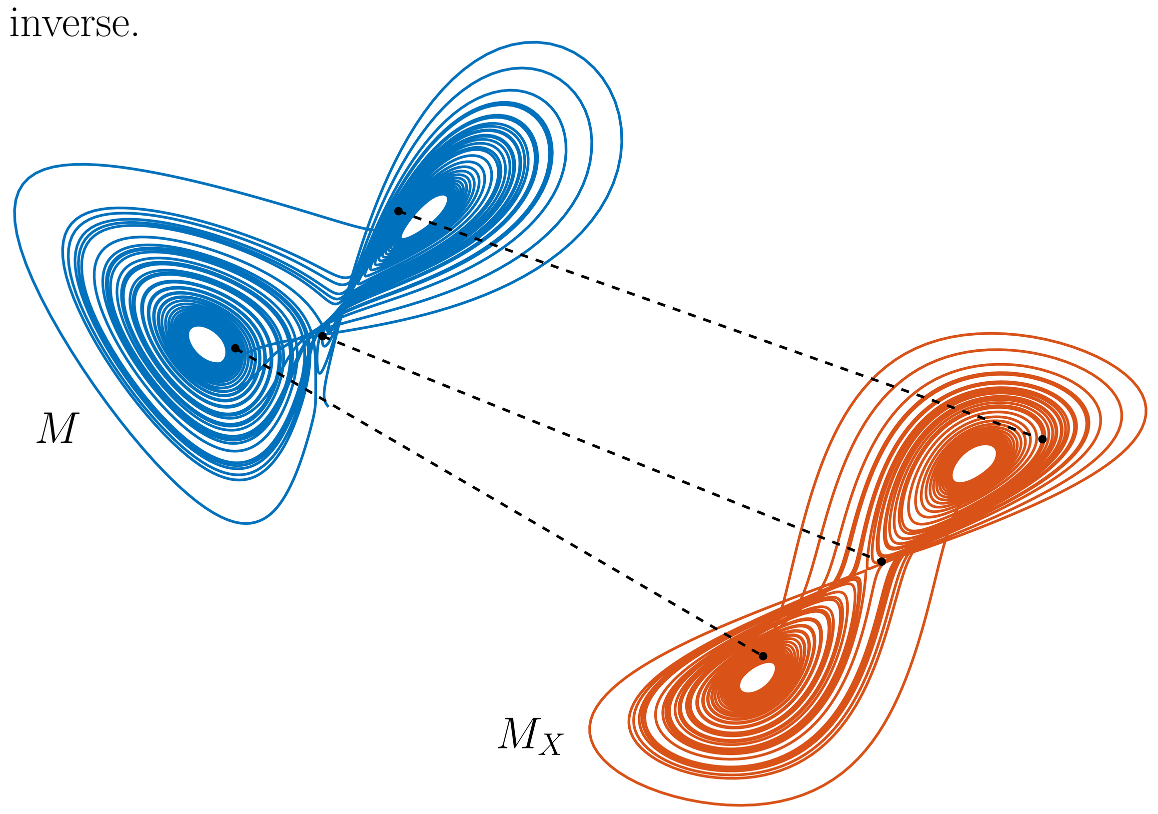


The shadow manifold of X, M_X:

$$M_X = (X(t), X(t-\tau), \cdots, X(t-(E-1)\tau))$$

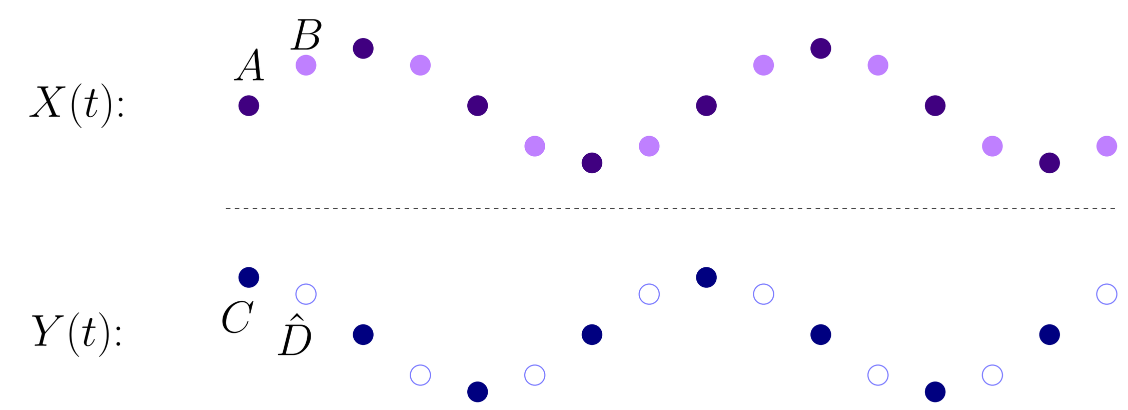


Takens's Theorem: M_X and M are diffeomorphic, i.e., there exists a differentiable bijection between the two with a differentiable inverse.



Split Convergent Cross Mapping (sCCM)

Our Model: Split Convergent Cross Mapping for $\kappa = 2$



1. Split $X(t)$ into A and B and $Y(t)$ into C and D
2. Create M_A, M_B, M_C, M_D by plotting time lags
3. Estimate $\hat{D}$:
   1. For each point in M_B, find the nearest neighbors in M_A
   2. Compute weights of the nearest neighbors in M_A
   3. Apply weights to corresponding points in M_C to get $M_{\hat{D}}$
   4. Average dimensions of $M_{\hat{D}}$ to get $\hat{D}$

   Repeat similar procedure for others. For $\kappa > 2$, consider each pair of splits together, and reconstruct each split from all others, forming $\kappa - 1$ reconstructions, averaging to get one reconstruction.
4. Combine all the estimations to get $\hat{X}$ and $\hat{Y}$

Takens's theorem provides an explanation for why sCCM may decompose signals. Consider a composite system with only two constituents, $(X, Y) = (X_1, Y_1) + (X_2, Y_2)$. Through sCCM, we construct a linear mapping T_M from M_Y to M_X. We observe that the construction of T is influenced more by X_1 and Y_1, so T_M is close to a true diffeomorphism from M_{Y_1} to M_{X_1}. In practice, if (X_1, Y_1) and (X_2, Y_2) are different enough, it is unlikely this diffeomorphism maps M_{Y_2} to any relevant manifold. With this, we can create a linear mapping T by composing T_M with the mappings to and from the shadow manifolds. Thus,

$$\hat{X} = TY = TY_1 + TY_2 = X_1 + \text{noise}$$
$$X - \hat{X} = X - TY = X_2 + \text{noise}.$$

Testing sCCM

Consider two causally related time series, $X(t)$ and $Y(t)$, formed by the combination of $n \geq 2$ distinct dynamical systems such as the Lorenz, Rossler, and Chen systems, i.e.,

$$(X, Y) = (X_1, Y_1) + \cdots + (X_n, Y_n),$$

where the terms (X_i, Y_i) are in order of decreasing amplitude. We apply sCCM on (X, Y) to get the reconstructed signals $\hat{X}(t)$ and $\hat{Y}(t)$ which through observation approximate the dynamics of the dominant system (X_1, Y_1), sharing a diffeomorphism between them. Thus, lower amplitude systems (interference) can be isolated by computing:

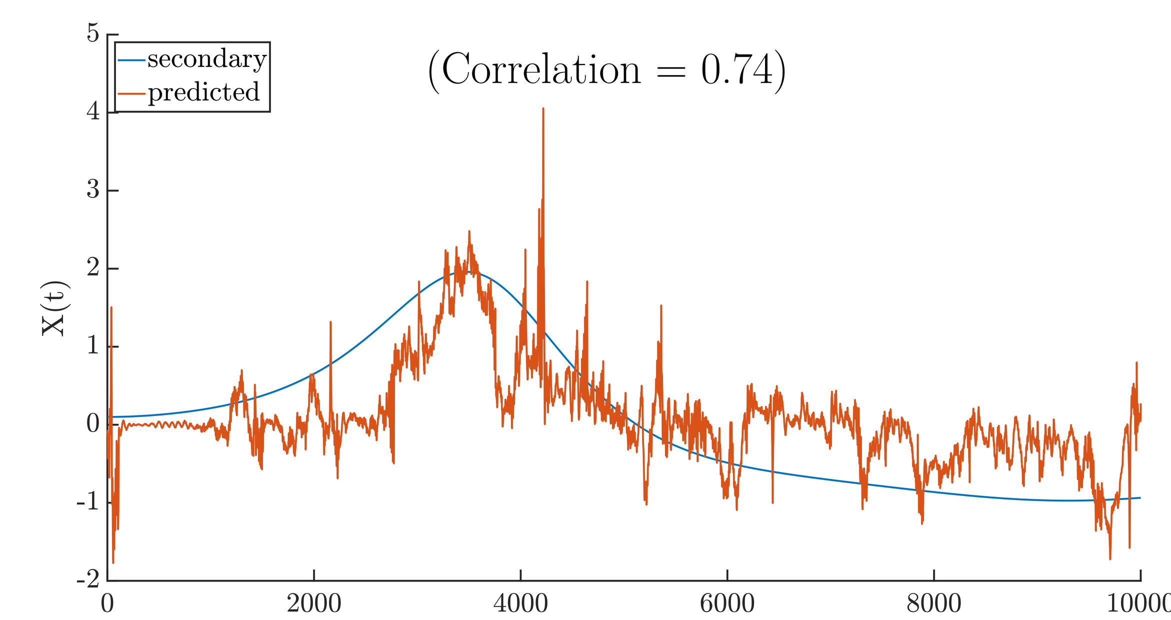
$$X_{\text{resids}}(t) = X(t) - \hat{X}(t) \approx X_2(t) + \cdots + X_n(t)$$
$$Y_{\text{resids}}(t) = Y(t) - \hat{Y}(t) \approx Y_2(t) + \cdots + Y_n(t).$$

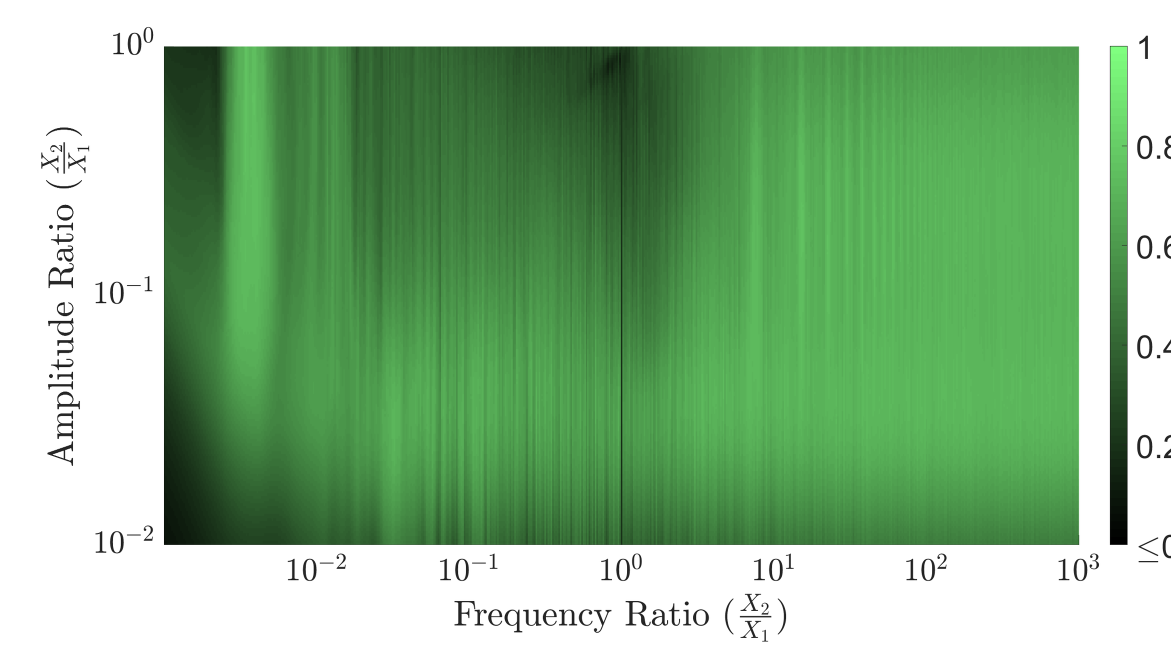
To evaluate performance, the Pearson correlation between the reconstructed signal $\hat{X}_i$ and the true signal X_i is computed. We first test our method on simpler chaotic systems and later apply it to real Hall thruster data. We use the Fast Fourier Transform as a benchmark for comparison.

Choosing Parameters

- Time Lag τ – time delay of phase space: Average Mutual Information
- Embedding Dimension E – dimension of the reconstruction of the shadow manifold: Cao's Method
- Number of Nearest Neighbors – number of points used to reconstruct each point in shadow manifold: grid search
- Number of Splits κ – number of parts to split time series: dependent on density of time series

Results: Test Data



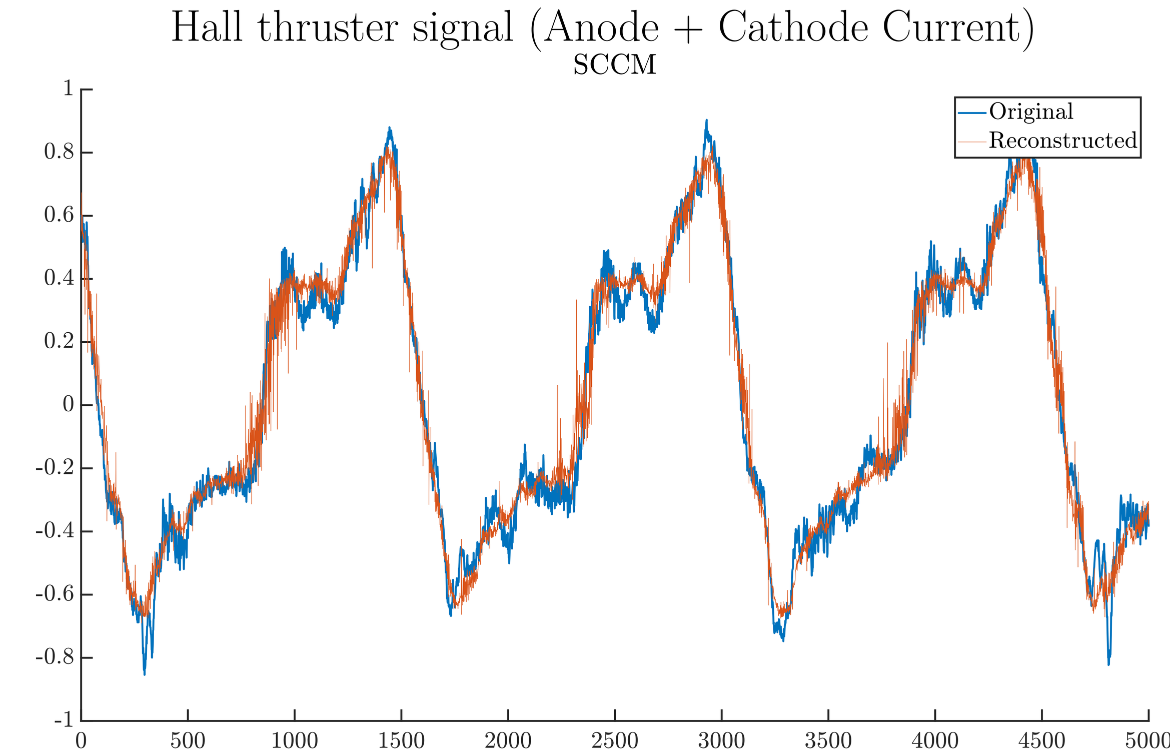


Applying sCCM to composite systems of two Lorenz systems yields high correlation values (0.99 correlation, on average) between the reconstructed sCCM signal $(\hat{X}(t), \hat{Y}(t))$ and primary Lorenz signal (X_1, Y_1) for almost all amplitudes and frequencies of interference signals tested. The above plots depict the time series of reconstructed secondary signal and true secondary signal for a fixed frequency and amplitude and correlations between X_{resids} with X_2 for a composite system of two Lorenz systems for various frequencies and amplitudes, respectively. The average correlation value for the secondary signals was 0.565 for the amplitudes and frequencies tested.

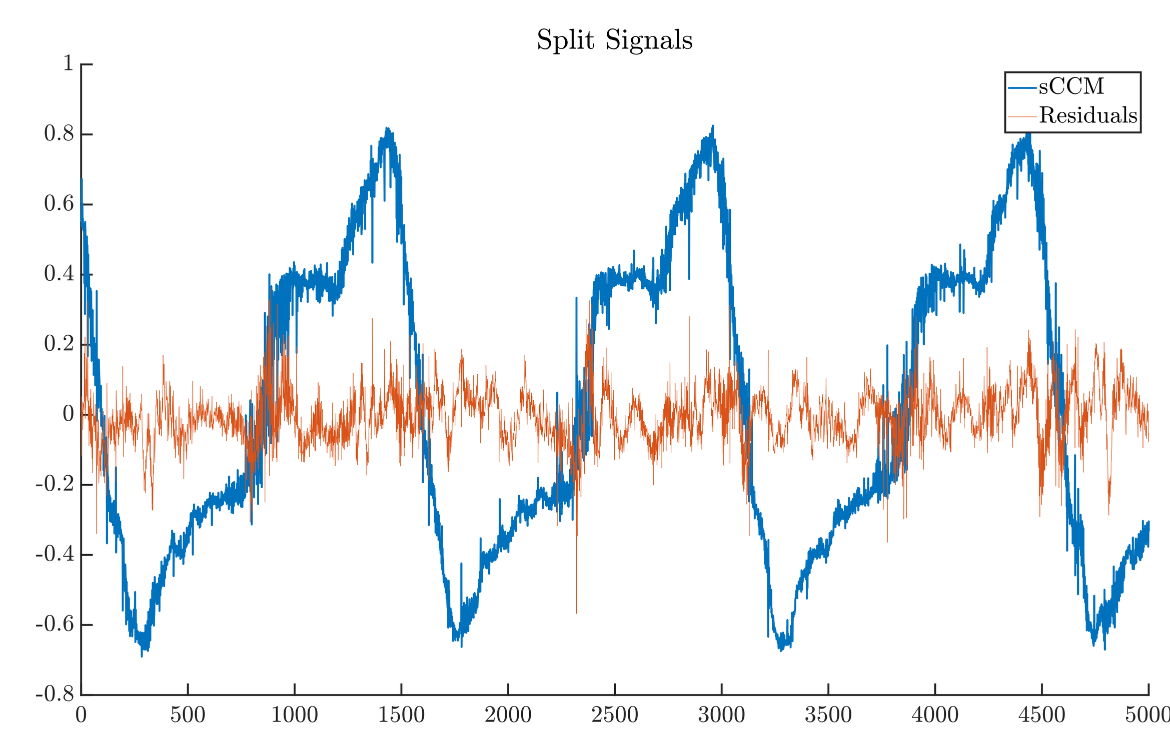
Results: Hall Thruster Data

The following plots show the results of sCCM when reconstructing the sum of the anode current and cathode current from the electrical signal at a ring from behind the Hall thruster, and then normalized relative to the original signal. The two reconstructions are clearly distinct time series: the primary reconstruction is a large amplitude and low frequency quasiperiodic signal, while the secondary reconstruction is a small amplitude and high frequency signal or noise.

Hall thruster signal (Anode + Cathode Current)



Decomposed Hall thruster Signal via sCCM



Discussion and Future Work

Empirically, sCCM works well on decomposing composite Lorenz and Rossler systems of various frequencies and amplitudes. Applying sCCM to Hall thruster data, we find evidence that the signals obtained from sCCM represent underlying signals present in the system.

In practice, Hall thrusters are tested in simulated vacuum environments although when sent into space, Hall thruster signals are often corrupted by interfering signals; traditional methods such as the Fourier transform typically fail to decompose these chaotic signals. sCCM opens new pathways for improving AFRL's testing by revealing deviations between reconstructed residuals in a simulated environment and in space. Potential avenues for future research include investigating the optimal rotation and the effects of principal component analysis before applying sCCM and further exploring whether the residuals in Hall thruster data represent noise or an underlying signal.

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